Last time: - measure spaces

- Lebesgue measure

- integration

$[0, \frac{1}{2}]$? $\frac{1}{2} + (0, \frac{1}{2})$ ''

$[0,1] = \{0\} \cup (0, \frac{1}{2}) \cup \{\frac{1}{2}\} \cup (\frac{1}{2}, 1) \cup \{1\}$

$1 = 0 + x + 0 + x + 0$

$x = \frac{1}{2}$

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Recall:

A simple function $X \to \mathbb{R}$ is a linear combination of indicator functions $\mathbb{1}_{E_i}$ of measurable sets E_i.

$$\mathrm{Sim}_X \doteq \mathrm{Span}_{\mathbb{R}} \{ \mathbb{1}_E \mid E \in \mathcal{M} \}.$$

If $f : X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ then we write

$$\mathrm{Sim}(f) \doteq \{ \phi \leq f \mid \phi \in \mathrm{Sim}_X \}$$

We define integration of simple functions by

$$\int \mathbb{1}_E \, d\mu = \mu(E)$$

and extending by linearity to Sim_X

(checking that this is well-defined)

Now given any measurable function $f : X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ we define

$$\int f \, d\mu \doteq \sup \int \mathrm{Sim}(f)$$

where $\int \mathrm{Sim}(f) \doteq \{ \int \phi \, d\mu \mid \phi \in \mathrm{Sim}(f) \}$

We say that f is integrable if

$$\int |f| \, d\mu < \infty .$$

Lemma: Let $f, g : X \to \mathbb{R}$ be measurable fns.

1) If $\{x \in X \mid f(x) \neq g(x)\}$ has measure 0, then $\int f \, d\mu = \int g \, d\mu$

2) If $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$

Assume f, g are integrable

3) $\int af \, d\mu = a \int f \, d\mu$ for all $a \in \mathbb{R}$

4) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Pf: Just consider when $f, g, a \geq 0$

2) $Sim(f) \subseteq Sim(g)$

$$\Rightarrow \int \mathrm{Sim}(f) \subseteq \int \mathrm{Sim}(g)$$

$$\Rightarrow \int f \, d\mu = \sup \int \mathrm{Sim}(f) \leq \sup \int \mathrm{Sim}(g) = \int g \, d\mu$$

3) $\mathrm{Sim}(af) = a\,\mathrm{Sim}(f)$

4) $\mathrm{Sim}(f+g) = \mathrm{Sim}(f) + \mathrm{Sim}(g)$

$$= \{ \phi_1 + \phi_2 \mid \phi_1 \in \mathrm{Sim}(f), \phi_2 \in \mathrm{Sim}(g) \} \qquad \square$$

Lem (Fatou lemma):

Let (f_n) be a sequence of measurable functions $f_n : X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Then

$$\int \liminf_{n \to \infty} f_n(x) \, d\mu(x) \leq \liminf_{n \to \infty} \int_X f_n \, d\mu$$

Lem: $\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$

Hint: First prove that

$$\limsup_{n\to\infty} \int |f - f_n| \, d\mu \leq \int \limsup_{n\to\infty} |f - f_n| \, d\mu$$

Thm (Dominated convergence):

If (f_n) is a sequence of measurable functions $X \to \mathbb{R}$ which converges pointwise to a function f, and there is an integrable function $g : X \to \mathbb{R}_{\geq 0}$ such that $|f_n| \leq g$ for all n, then f is integrable and

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

Pf: Note $|f_n - f| \leq 2g$

Apply Fatou to $2g - |f_n - f|$

$$\int \liminf_{n\to\infty} (2g - |f_n - f|)\, d\mu \leq \liminf_{n\to\infty} \int (2g - |f_n - f|)\, d\mu$$

$$\int 2g \, d\mu \leq \int 2g\, d\mu - \limsup_{n\to\infty} \int |f_n - f|\, d\mu$$

$$\limsup_{n\to\infty} \int |f_n - f|\, d\mu \leq 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \int |f_n - f|\, d\mu = 0.$$

$\square$

Def: Let $p \geq 1$.

The **p-norm** is defined for measurable $f : X \to \mathbb{R}$ by

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

Define $\mathcal{L}^p(X) \doteq \{ f : X \to \mathbb{R} \text{ measurable} \mid \|f\|_p < \infty \}$.

$$L^p(X) \doteq \mathcal{L}^p / \{ f \in \mathcal{L}^p \mid \|f\|_p = 0 \}.$$

Thm: $L^p(X)$ is a Banach space with the p-norm.

Thm: $L^2(X)$ is a Hilbert space.